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Topological electromagnetism

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Abstract. There is a topological structure in the set of the electromagnetic radiation fields (with $\boldsymbol{E}\cdot\boldsymbol{B}=0$) in vacuum. A subset of them, called here *the admissible fields*, are associated with maps $S^3 \mapsto S^2$ and can be classified in homotopy classes labelled by the value of the corresponding Hopf indexes, which are topological constants of the motion. Moreover, any radiation field can be obtained by patching together admissible fields and is therefore locally equal to one of them. There is, however, an important difference from the global point of view, since the admissible fields obey the topological quantum conditions that the magnetic and the electric helicities are equal to integer numbers n and m times an action constant a which must be introduced because of dimensional reasons, that is $\int \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}^3r = na$, $\int \boldsymbol{C}\cdot\boldsymbol{E}\,\mathrm{d}^3r = ma$, where $\boldsymbol{B}$ and $\boldsymbol{E}$ are the magnetic and electric fields and $\nabla\times\boldsymbol{A}=\boldsymbol{B}$, $\nabla\times\boldsymbol{C}=\boldsymbol{E}$. A topological mechanism for the quantization of the electric charge operates in the set of the admissible fields, in such a way that the electric flux through any closed surface around a point charge is always equal to $\sqrt{a}$ times an integer number n', equal to the degree of a map $S^2 \mapsto S^2$, corresponding to the existence of a fundamental charge with value $q_0 = \sqrt{a}/4\pi$. It is argued that results of this kind could help reaching a better understanding of quantum physics.

1. Introduction

There is little doubt that topology will in future play a major role in quantum physics. As Atiyah (1990) puts it, this is not surprising, since 'both topology and quantum physics go from the continuous to the discrete'. Topological ideas were already used in the study of the structure of matter more than a century ago by Lord Kelvin, who imagined in 1868 that the atoms could be knots or links of the vorticity lines of the aether, to which he applied the then new Helmholtz theorems on fluid dynamics (Kelvin 1868, Tait 1911, Archibald 1989). He understood, in a remarkable combination of geometrical insight and physical intuition, that such knots and links would be extremely stable, just as matter is. Furthermore, the many different ways in which curves can be linked or knotted offered an explanation for the remarkable variety of the properties of the chemical elements. From our modern perspective, we can add to stability and variety two other very important qualities of matter which were not known in Kelvin's time. One is transmutability, the ability of atoms to change into others of a different kind as an effect of nuclear reactions, which could be related to the breaking and reconnections of vorticity lines (as happens, for instance, to the magnetic lines in tokamaks during disruptions). The second is the discrete energy spectrum, which is also a characteristic of the non-trivial topological configurations of vector fields, as has been proved by Moffatt (1990a), making use of a theorem by

Freedman (1988). At the time, Kelvin's model had a good reception, being praised, for instance, by Maxwell. But neither topology nor atomic phenomenology were sufficiently developed to follow his very deep insight, which explains why it was later forgotten and remained unknown for a very long time.

Topology was also at the basis of one of Dirac's (1931) most appealing and intriguing proposals, the monopole, which embodied a mechanism for the quantization of the electric charge, an idea developed later in other contexts (Polyakov 1974, 't Hooft 1974). Since Aharonov and Bohm (1959) discovered the effect that bears their name, it has been known that the description of the electromagnetic field requires topological considerations. The sine–Gordon equation offers the simplest model with a conservation law of topological origin, based on the degree of a map $S^1 \mapsto S^1$ of the circle on itself. Its extension to three dimensions allowed Skyrme (1961, 1988) to build a model with topological solitons and conserved current, the corresponding quantity taking only integer values equal to the degree of a map $S^3 \mapsto S^3$ between three-dimensional spheres. As Skyrme explained he had three motives for proposing such a model: unification; renormalization; and what he called the fermion problem. His skyrmion, as his basic soliton became known, would be a fundamental boson from which he hoped to build all the particles and, because a topological theory must be nonlinear, the possibility of removing the infinities seemed a real and attainable aim.

In the past, the applications of topology to field theory concentrated on knots and links (Atiyah 1990), the classification of which had been attempted by Tait (1911) after discussions with Kelvin concerning his atomic models. He was then able to pose the problem and to formulate some conjectures. However, in spite of its interest and beauty, this branch of mathematics fell into oblivion for several decades, until the discovery in 1928 of the Alexander polynomials, which are invariants associated with each knot or link. In 1984, Jones found another set of polynomials, which were found to be very useful for classifying knots or links, which allowed proof of some of Tait's conjectures. Although these developments arose from pure mathematics, they turned out to be related to Yang–Mills field theory, a very important physical application being the construction of a topological quantum field theory, as proposed by Witten (1988, 1989), which may open the way to a deeper understanding of quantum physics.

It will be shown in this paper that the set of the radiation solutions of Maxwell equations in vacuum (those which verify $\boldsymbol{E} \cdot \boldsymbol{B} = 0$) has a subset, called here *the admissible fields* with a curious topological structure shown in the links formed by pairs of field strength lines. Any pair of maps $\phi, \theta : S^3 \mapsto S^2$ (with orthogonal level curves) is associated with an admissible electromagnetic field, such that the magnetic and electric lines are the level curves of ϕ and θ, respectively. As a consequence, the linking numbers of any pair of magnetic lines and of any pair of electric lines are two topological constants of the motion, taking only integer values n and m, equal to the Hopf index of the corresponding maps. Moreover, they are also equal to the magnetic and electric helicities $\int \boldsymbol{A}\cdot\boldsymbol{B}\, \mathrm{d}^3r = n$, $\int \boldsymbol{C}\cdot\boldsymbol{E}\, \mathrm{d}^3r = m$, where $\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{B}$, $\boldsymbol{\nabla}\times\boldsymbol{C} = \boldsymbol{E}$. (In natural units, otherwise the helicities would be equal to na and ma respectively, a being an action constant which should be introduced because of dimensional reasons.) The admissible fields can thus be classified in homotopy classes labelled by the integer values of both helicities. The potential importance of this property rests on the fact that any standard electromagnetic radiation field is locally equal to an admissible one, except in a set of zero measure, and can be obtained by patching together several admissible fields. This can be expressed by saying that the difference between the set of admissible fields and that of all the solutions of Maxwell equations in vacuum, verifying $\boldsymbol{E} \cdot \boldsymbol{B} = 0$, is not local but global.

In a model based on the admissible fields, the electric charge is quantized since that of a point particle is neccesarily an integer multiple of the fundamental value $1/4\pi$ (in physical units $\sqrt{a}/4\pi$). The topological ground for this property is that, in this model, the flux through a sphere which encloses a charge is equal to the degree of a map $S^2 \mapsto S^2$ between two spheres which is always an integer number.

The plan of the paper is as follows. The theory of the Hopf index is summarized in section 2. In section 3, the set of admissible fields is described and a model based on them is presented. Section 4 is devoted to the relation between the set of all the solutions of Maxwell equations in vacuum and the subset of the admissible fields. In section 5, the quantization of the helicity is considered, as well as the relation between the magnetic helicity and the projection of the spin on the direction of motion. In section 6, the topological mechanism for the quantization of the electric charge is presented. Finally, in section 7, the results are summarized.

2. The Hopf index

Let us consider a scalar field $\phi(\boldsymbol{r})$ with the property that its value at infinity is well defined, which means that its limit when $r \to \infty$ does not depend on the direction. In that case, it can be interpreted as a map $\phi : S^3 \mapsto S^2$, after identifying, via stereographic projection, $R^3 \cup \{\infty\}$ with the sphere S^3 and the complete complex plane $C \cup \{\infty\}$ with the sphere S^2.

These identifications can be realized in the following way. A point P in S^2 can be represented by: (i) the cartesian n_k, such that $n^2 = \sum n_k^2 = 1$; or (ii) the spherical ϑ, φ, related to n_k by $n_3 = \cos\vartheta$, $n_1 + \mathrm{i}n_2 = \sin\vartheta \exp(\mathrm{i}\varphi)$. Its stereographic projection is the complex number $\phi = \cot(\vartheta/2)\exp(\mathrm{i}\varphi)$, which will be taken in this work as the coordinate of P in S^2. On the other hand, the cartesian $u_1, \ldots, u_4$ with the condition $\sum u_k^2 = 1$ can be taken as coordinates of a point Q in S^3. Their relations with the cartesian $x_k, k = 1,2,3$ of the stereographic projection on the three-space $u_4 = 0$ are $x_k = u_k/(1 - u_4), u_k = 2x_k/(1 + r^2)$, $r^2 = \sum x_k^2, k = 1,2,3, u_4 = (r^2 - 1)/(r^2 + 1)$.

This explains why and how a complex function $\phi(\boldsymbol{r})$ can be interpreted as a map $S^3 \mapsto S^2$. As maps of this kind can be classified in homotopy classes, labelled by a topological invariant called the Hopf index, the same property applies to complex scalar fields.

Given a map $f : S^3 \mapsto S^2$, which for simplicity will be taken to be smooth, the inverse image of any two points a and b of S^2, $f^{-1}(a)$ and $f^{-1}(b)$, are in general two closed curves in S^3. Its linking number is defined as the number of intersections of any of the two with an oriented surface bounded by the other one. It does not depend on the particular pair of points, since by moving continuously from (a, b) to (a', b') the inverse images can neither untie nor tie further one to the other (since, for this to happen, they should have a common point with two different images). Furthermore and for the same reason, if the map f evolves continuously in time this number must be constant. The set of maps $S^3 \mapsto S^2$ can thus be classified in homotopy classes, each one labelled by an integer number n. This number is called the Hopf index (Hopf 1931, Nicole 1978, Bott and Tu 1982, Kundu and Rybakov 1982, Nash and Sen 1982, Kundu 1982, 1986, Moffatt 1990b).

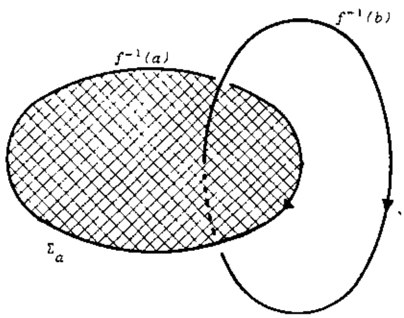

Figure 1. The closed curves $f^{-1}(a)$ and $f^{-1}(b)$.

Let us take the two inverse images of a and b as in figure 1 and let Σ_a be a surface bounded by $f^{-1}(a)$, which is noted as $\partial\Sigma_a = f^{-1}(a)$. It is clear that the number n of intersections of $f^{-1}(b)$ with Σ_a, is equal to the degree of the restriction of the map f to Σ_a (because each point of S^2 has n inverse images in this surface). The degree of such a map is equal to the integral over Σ_a of the area 2-form of S^2 normalized to unity. This area 2-form, expressed in stereographic coordinates, is equal to

$$\mathcal{F} = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}\phi^* \wedge \mathrm{d}\phi}{(1+\phi^*\phi)^2}. \tag{1}$$

(To simplify the notation the symbol $\mathcal{F}$ is used both for the area 2-form and its pull-back to S^3 or R^3.) Since $\mathcal{F}$ is closed in S^3 whose second group of cohomology is trivial, it must also be exact or, in other words, there exists a 1-form $\mathcal{A}$ such that $\mathcal{F} = \mathrm{d}\mathcal{A}$. As was shown in 1947 by Whitehead, the integral of $\mathcal{F}$ through Σ_a which gives the Hopf index can then be written as

$$n = \int_{S^3} \mathcal{A} \wedge \mathcal{F}. \tag{2}$$

The area 2-form being much used in the following, this argument must be presented in more detail. The pull-back of this form to R^3 by the map $\phi(\boldsymbol{r})$ can be written as

$$\mathcal{F} = \frac{1}{2} F_{ij}\, \mathrm{d}x_i \wedge \mathrm{d}x_j = \frac{1}{4\pi \mathrm{i}} \frac{\partial_i\phi^*\partial_j\phi - \partial_j\phi^*\partial_i\phi}{(1+\phi^*\phi)^2}\, \mathrm{d}x_i \wedge \mathrm{d}x_j. \tag{3}$$

Like any antisymmetric tensor, F_{ij} can be expressed in terms of a vector $\boldsymbol{B}(\boldsymbol{r})$ as

$$F_{ij} = -\epsilon_{ijk} B_k \qquad B_k = -\tfrac{1}{2}\epsilon_{ijk} F_{ij} \tag{4}$$

and, as it follows easily from (3), it turns out that $\boldsymbol{B}$ is divergenceless ($\nabla \cdot \boldsymbol{B} = 0$). Consequently, the forms $\mathcal{F}$ and $\mathcal{A}$ are expressed in R^3 as

$$\mathcal{F} = -\tfrac{1}{2}\epsilon_{ijk} B_k\, \mathrm{d}x_i \wedge \mathrm{d}x_j \qquad \mathcal{A} = -A^i \mathrm{d}x_i \tag{5}$$

where $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. It is clear that the vector $\boldsymbol{B}$, which plays an important role in the description of the maps from S^3 (or R^3) to S^2 (or C), is always tangent to the level curves of ϕ, which are its integral lines. It will be called here *the Whitehead vector* of the map ϕ and noted $\boldsymbol{B} = \boldsymbol{W}(\phi)$. The equations (3)–(5) show that a structure quite similar to a magnetic field, or with an electric field in vacuum, is associated to any map $S^3 \mapsto S^2$. The expression (2) of the Hopf index can then be written as

$$n = \int_{R^3} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}^3 r. \tag{6}$$

The quantity in the right-hand side of (6) has been used in physics in several contexts and is called *the helicity* of the vector $\boldsymbol{B}$. A discussion of its properties will be published elsewhere (Rañada 1992). It was first used by Woltjer (1958) to obtain the stable configurations of astrophysical plasmas, $\boldsymbol{A}$ and $\boldsymbol{B}$ being the vector potential and the magnetic field. The term helicity was coined by Moffatt (1969) in a paper on tangled vorticity lines, with the velocity of a fluid $\boldsymbol{v}$ and its vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ as the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ (Moffatt 1969, 1981, 1990b, Moffatt and Tsinober 1990, Bajer and Moffatt 1990). Later it was applied to the magnetic relaxation of plasmas in toroidal vessels (Taylor 1974, 1986, Pfister and Gekelman 1991). Kuznetsov and Mikhailov (1980), following previous work by Faddeev (1976), established the relation between mathematics and physics by proving that it can be identified in some cases with the Hopf invariant. Its value can also be expressed as

$$h = n\gamma^2 \tag{7}$$

where n is the linking number, as defined earlier, and γ is the total strength of the field, that is the sum of the strengths of all the tubes formed by the integral lines of $\boldsymbol{B}$ (the magnetic lines, for instance) (Rañada 1992). This means that a non-zero value of the helicity is a sure indication of non-trivial topology of the integral lines of a vector field. In some situations, as in astrophysics or in toroidal vessel plasmas, the relaxation consists in the rapid decreasing of the energy to the minimum value compatible with the consevation of the magnetic helicity. This property is also important in fluids and allows discrete energy spectra to be assigned to configurations characterized by non-trivial configurations of vorticity (Moffatt 1990a, Freedman1988).

3. A topological model of the electromagnetic radiation fields

The considerations of the previous section suggest that there is a formal relation between the set of maps $S^3 \mapsto S^2$ and that of the electromagnetic fields (Rañada 1989). It will now be shown that this is indeed the case, to such an extent that a model of electromagnetic radiation fields in vacuum can be constructed in which the electric and the magnetic fields are the Whitehead vectors of two maps $S^3 \mapsto S^2$, given by two complex scalar fields $\phi(\boldsymbol{r})$ and $\theta(\boldsymbol{r})$, their level curves being the magnetic and electric lines, respectively. In our notation $\boldsymbol{B} = \boldsymbol{W}(\phi)$, $\boldsymbol{E} = \boldsymbol{W}(\theta)$. These scalars will vary in time and, because of evident covariance reasons, it follows from (3) that the electromagnetic (or Faraday) tensor $F_{\mu\nu}$ and its dual (or Maxwell) tensor $M_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ have the value

$$F_{\mu\nu} = f_{\mu\nu}(\phi) = \frac{1}{2\pi \mathrm{i}} \frac{\partial_\mu \phi^* \partial_\nu \phi - \partial_\nu \phi^* \partial_\mu \phi}{(1 + \phi^* \phi)^2} \tag{8}$$

$$M_{\mu\nu} = f_{\mu\nu}(\theta) = \frac{1}{2\pi \mathrm{i}} \frac{\partial_\mu \theta^* \partial_\nu \theta - \partial_\nu \theta^* \partial_\mu \theta}{(1 + \theta^* \theta)^2} \tag{9}$$

it must be remarked that with this definition the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are inverse square lengths, in full agreement with what we know to be their natural dimensions. If one prefers instead to use physical units, a factor $\sqrt{a}$ should be introduced in (8)–(9), a being an action constant which gives the equivalence between the physical and the natural units and which fixes the scale of the topological constant of the motion. Moreover ϕ must be a scalar and θ a pseudoscalar. The electric and the magnetic fields are then related to (8) and (9) by

$$E_i = F_{0i} = \tfrac{1}{2}\epsilon_{ijk}M_{jk} \qquad B_k = -\tfrac{1}{2}\epsilon_{kij}F_{ij} = M_{0k}. \tag{10}$$

Notice that the corresponding 2-forms $\frac{1}{2}F_{ij}\,\mathrm{d}x_i \wedge \mathrm{d}x_j$, $\frac{1}{2}M_{ij}\,\mathrm{d}x_i \wedge \mathrm{d}x_j$ (remember that they are equal to the area 2-forms pulled back from the sphere S^2 by the maps ϕ and θ, respectively) are the magnetic and electric flux forms, respectively.

The procedure follows with the Lagrangian density

$$L = -\tfrac{1}{8}\left(F_{\mu\nu}F^{\mu\nu} + M_{\mu\nu}M^{\mu\nu}\right) \tag{11}$$

submitted to the duality condition or constraint

$$G_{\alpha\beta} = M_{\alpha\beta} - \tfrac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu} = 0. \tag{12}$$

According to the method of the Lagrange multipliers, the modified Lagrangian density

$$L' = L + \mu^{\alpha\beta}G_{\alpha\beta} \tag{13}$$

must be used, the multipliers being the components of the constant tensor $\mu^{\alpha\beta}$. A simple calculation shows that the constraint (12) does not contribute to the Euler–Lagrange equations, which happen to be

$$\partial_\alpha F^{\alpha\beta}\partial_\beta\phi = 0 \qquad \partial_\alpha F^{\alpha\beta}\partial_\beta\phi^* = 0 \tag{14}$$

$$\partial_\alpha M^{\alpha\beta}\partial_\beta\theta = 0 \qquad \partial_\alpha M^{\alpha\beta}\partial_\beta\theta^* = 0 \tag{15}$$

the consequence being that, if the Cauchy data $(\phi, \partial_0\phi, \theta, \partial_0\theta)$ at $t = 0$ verify the constraint (12), it will be maintained for all $t > 0$. It follows then that both $F_{\alpha\beta}$ and $M_{\alpha\beta}$ satisfy the Maxwell equations in vacuum. In fact, the first pair for both tensors

$$\epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta} = 0 \qquad \epsilon^{\alpha\beta\gamma\delta}\partial_\beta M_{\gamma\delta} = 0 \tag{16}$$

holds authomatically for any solution (ϕ, θ), because of the definitions (8) and (9). On the other hand, as $F_{\mu\nu}$ and $M_{\mu\nu}$ are dual to each other, it follows from (12) and (16) that

$$\partial_\alpha F^{\alpha\beta} = 0 \qquad \partial_\alpha M^{\alpha\beta} = 0 \qquad \beta = 0, 1, 2, 3, \ldots$$

which is the second Maxwell pair for the two tensors. In other words, *if ϕ and θ obey the Euler–Lagrange equations* (14) *and* (15), *then $F_{\alpha\beta}$ and $M_{\alpha\beta}$, as defined by* (8) *and* (9), *verify the Maxwell ones and are, therefore, electromagnetic fields of the standard theory, provided that they are dual to each other at time $t = 0$.* The reason is that the Maxwell equations in vacuum have the property that, if two dual antisymmetric tensors verify the first pair, they also verify the second.

However, a warning is necessary. It is clear from definitions (8) and (9) that the electromagnetic fields verify in this model the condition $\boldsymbol{E} \cdot \boldsymbol{B} = 0$, i.e. the electric and the magnetic fields are orthogonal to each other. In other words, the topological structure which is used to build this model contains only a subset of the solutions of the Maxwell equations in vacuum. This limits the scope of the model, although it includes two very important cases: free radiation fields and the Lienard–Wiechert potential for a charge, which will be treated in sections 4 and 6. This question can be posed in the following way. According to the Darboux theorem (Godbillon 1969, Choquet-Bruhat *et al* 1982) the Faraday form can always be written as

$$\mathcal{F} = \mathrm{d}q_1 \wedge \mathrm{d}p^1 + \mathrm{d}q_2 \wedge \mathrm{d}p^2 \tag{18}$$

where q_k, p^k are four functions in spacetime. Note that each one of the two terms in (18) verifies the condition $\boldsymbol{E} \cdot \boldsymbol{B} = 0$, although this is not the case of their sum. However, this representation is not unique, since we can make canonical transformations to new variables $q_k, p^k \to Q_k, P^k$ without changing the functional form of (18) (Goldstein 1980, Misner *et al* 1973). In physical terms, this can be understood in the following way. Take an electromagnetic field with Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}$. By a suitable Lorentz transformation (with direction $\boldsymbol{n}$ and velocity parameter η given by $\boldsymbol{n} \tanh 2\eta = 2\boldsymbol{E} \times \boldsymbol{B}/(E^2 + B^2)$) (Misner *et al* 1973), we can change to a frame in which $\boldsymbol{S} = 0$ at any prescribed point P, which means that $\boldsymbol{E}$ and $\boldsymbol{B}$ are there parallel. Taking their common direction as the Oz-axis, the Faraday form can be written as in (18)

$$\mathcal{F} = \mathrm{d}t \wedge \mathrm{d}(Ez) + \mathrm{d}(Bx) \wedge y \tag{19}$$

(because $\mathcal{F}$ is closed). In general, the Faraday 2-form (18) cannot be expressed in a simpler manner, because it is of rank four and also of class four (i.e. four 1-forms and four functions are needed to express it, respectively). However, there are important cases in which a simpler representation is possible, as happens for instance for radiation fields in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are orthogonal. If this is so, the Faraday form is degenerate, of only rank two and class two, since it can be written by using only two functions $p(\boldsymbol{r}, t)$, $q(\boldsymbol{r}, t)$ and two 1-forms $\mathrm{d}p, \mathrm{d}q$ as

$$\mathcal{F} = \mathrm{d}q \wedge \mathrm{d}p. \tag{20}$$

It is then said that the field is singular because, as $\det(F_{\mu\nu}) = (\boldsymbol{E} \cdot \boldsymbol{B})^2$, the matrix of the electromagnetic tensor is singular (notice that this means that it has no inverse, not that their components have singularities as functions in the spacetime). Singular fields are very important, since they include free radiation fields, the Lienard–Wiechert potentials for one particle and the pure electric or pure magnetic fields. In its present form, this model applies only to singular fields. It is easy to understand why, just by looking at the area 2-form (1) which is our basic element, since if $\phi = S \exp(\mathrm{i}2\pi\sigma)$ it can be expressed as (20) with $q = -1/(1 + S^2), p = \sigma$, from which we see that it is degenerate. To avoid the confusion which the term singular might bring, we will speak instead of radiation fields.

It is important to emphasize, furthermore, that the topological structure on which this model is based contains only radiation fields of a very special kind since they all can be deduced from scalars according to equations (8)–(9), a very special property. They are thus a subset of the set of the solutions of the Maxwell equations in vacuum. We will say that they are the *admissible fields*. However, as will be shown in the rest of this section and in the next, they have other curious properties which justify their detailed study.

3.1. The Cauchy data

The Cauchy data, $(\phi(\boldsymbol{r},0), \partial_t\phi(\boldsymbol{r},0), \theta(\boldsymbol{r},0), \partial_t\theta(\boldsymbol{r},0))$, must be characterized. As was shown earlier, if they verify the duality condition (12), the solution satisfies it for any time $t > 0$. In this case, both the Cauchy data and the corresponding solutions will be admissible. This poses the problem of establishing whether there are admissible data and solutions, although the answer is quickly seen to be positive, because the duality condition can be understood as a set of six real partial differential equations for eight real functions (the real and imaginary parts of the Cauchy data). This is indeed the case, as shown in the following.

If the three vectors associated with the tensor $f_{\mu\nu}(\chi)$ deduced from a scalar χ through (8) and (9) are written

$$\mathcal{E}_i(\chi) = f_{0i}(\chi) \qquad \mathcal{B}_i = -\tfrac{1}{2}\epsilon_{ijk} f_{jk}(\chi) \tag{21}$$

the duality condition (12) takes the form

$$\mathcal{E}(\phi) = -\mathcal{B}(\theta) \qquad \mathcal{B}(\phi) = \mathcal{E}(\theta). \tag{22}$$

As the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ are orthogonal, the level curves of θ and ϕ must also be orthogonal. This can be written as

$$(\nabla\phi^* \times \nabla\phi)\cdot(\nabla\theta^* \times \nabla\theta) = 0 \tag{23}$$

which is a real partial differential equation for the two complex functions ϕ and θ and has an infinity of solutions. Each one gives a couple $\phi(\boldsymbol{r},0)$ $\theta(\boldsymbol{r},0)$. The time derivatives $\partial_t\phi(\boldsymbol{r},0)$, $\partial_t\theta(\boldsymbol{r},0)$ are then fixed by the duality condition (22). To see it, note that $\mathcal{B}(\theta)$ is a linear combination of $\nabla\phi^*$ and $\nabla\phi$, such as

$$\mathcal{B}(\theta) = b\nabla\phi^* + b^*\nabla\phi. \tag{24}$$

The function $b(\boldsymbol{r},0)$ is determined by $\phi(\boldsymbol{r},0)$ and $\theta(\boldsymbol{r},0)$, since

$$\mathcal{E}(\phi) = \frac{1}{2\pi\mathrm{i}}\frac{\partial_0\phi^*\nabla\phi - \partial_0\phi\nabla\phi^*}{(1+\phi^*\phi)^2} = -\mathcal{B}(\theta) \tag{25}$$

so that $\partial_0\phi = 2\pi\mathrm{i}b(\boldsymbol{r},0)(1+\phi^*\phi)^2$, an analogous expression holding for $\partial_0\theta$. Consequently, the Cauchy data consist of a pair of complex functions $\phi(\boldsymbol{r},0)$, $\theta(\boldsymbol{r},0)$ which verify condition (23), so that the system has two degrees of freedom with a differential constraint.

3.2. A hidden nonlinearity

We have here a structure with two levels. At the deeper one, it is nonlinear since the scalars ϕ and θ obey nonlinear equations. However the transformation T given by (8) and (9)

$$T : \phi \to F_{\mu\nu} = f_{\mu\nu}(\phi) \qquad \theta \to M_{\mu\nu} = f_{\mu\nu}(\theta) \tag{26}$$

changes these equations for ϕ and θ into the linear Maxwell ones, thus linearizing the model. This means that the standard electromagnetic equations can be derived from an underlying structure which is both nonlinear and topological. There is thus *a hidden nonlinearity*, shown by the fact that although the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ obey the linear Maxwell equations, not all its solutions are admissible, since they must verify the condition that the magnetic helicity be an integer number,

$$h_{\mathrm{mag}} = \int_{R^3} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}^3 r = n \tag{27}$$

where $\boldsymbol{A}$ is the vector potential. This is a topological quantum condition which do not allow new solutions to be obtained just by multiplying by any real number.

This curious situation is due to the fact that the transformation T of equation (26) is not invertible, because for the non-admissible solutions of the Maxwell equations $F_{\mu\nu}$, $T^{-1}(F_{\mu\nu})$ or $T^{-1}(M_{\mu\nu})$ are not defined, as explained earlier. This might seem disastrous, but as will be shown in the next section, the difference with Maxwell theory is only global because all the standard solutions are locally admissible, except in a zero measure set. There is indeed a difference with the standard theory, and an important one for that matter, but only if one looks globally to all the space R^3.

As the set of the admissible fields is not a linear space, the sum of two of them is not a new one, neither is it the product of one of them by a real number. However, there is still some linearity which shows in the following two properties.

Property 1. If $F_{\mu\nu}$ is admissible, all its integer multiples $nF_{\mu\nu}$ are also admissible.

It is easy to understand why. Let $\phi = S\exp(\mathrm{i}2\pi\sigma)$ and $\theta = R\exp(\mathrm{i}2\pi\rho)$. It is then clear that $nF_{\mu\nu}$ and $nM_{\mu\nu}$ are admissible since

$$nF_{\mu\nu} = f_{\mu\nu}(\phi^{(n)}) \qquad M_{\mu\nu} = f_{\mu\nu}(\theta^{(n)})$$

where $\phi^{(n)} = S\exp(n\mathrm{i}2\pi\sigma)$ and $\theta^{(n)} = R\exp(n\mathrm{i}2\pi\rho)$, which are well defined if n is an integer.

Property 2. If $F_{\mu\nu}$ is admissible and its scalars ϕ and θ never take the values 0 or ∞, all its real multiples $cF_{\mu\nu}$ are also admissible.

This is because, $cF_{\mu\nu}$ and $cM_{\mu\nu}$ are generated by $\phi^{(c)} = S\exp(c\mathrm{i}2\pi\sigma)$ and $\theta^{(c)} = R\exp(c\mathrm{i}2\pi\theta)$, respectively, which are well defined for real c. Remark that, in this case, ϕ and θ express maps $S^3 \mapsto R^1 \times S^1$ which form only one homotopy class. Notice also that the helicities vanish.

3.3. A topological quantization

As was stated earlier, all the solutions of the model verify a topological condition, expressed by (27), where n is the Hopf index of the map ϕ. As long as we remain in the vacuum, there is a second index since, for the same reasons, the electric helicity is quantized

$$h_{\mathrm{el}} = \int_{R^3} \boldsymbol{C} \cdot \boldsymbol{E} \, \mathrm{d}^3 r = m \tag{28}$$

where $\nabla \times \boldsymbol{C} = \boldsymbol{E}$. This implies that the electromagnetic fields of this model can be classified in homotopy classes, each one characterized by two integer numbers n, m. As a consequence and although it is classical in the sense that it only uses c-number fields, this model verifies a quantum condition and has a quantum-like character. Notice that we are using natural units, but, should we prefer to use physical ones, an action constant a should be needed because of dimensional reasons, since otherwise the Lagrangian density would not have the right dimensions (it would be the inverse of a length to the fourth power). In the definition of the tensors $F_{\mu\nu}$ and $M_{\mu\nu}$ the factor $\sqrt{a}$ should be introduced (since the non-natural dimensions of $\boldsymbol{E}$ and $\boldsymbol{B}$ are the square root of action divided by square of length), the form of the topological quantization being then

$$\int_{R^3} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}^3 r = na \qquad \int_{R^3} \boldsymbol{C} \cdot \boldsymbol{E} \, \mathrm{d}^3 r = ma \tag{29}$$

since, similar to the Planck factor $\hbar$, a is an action constant, this certainly reminds us of the quantization conditions of the old quantum theory

$$\oint p_k \, \mathrm{d} q_k = n_k h \tag{30}$$

or the property of quantum variables with a discrete spectrum which verify $A\psi_k = \alpha_k \hbar \psi_k$. This suggests that models of this kind might help us gain a better understanding of quantization.

4. The standard electromagnetic fields are locally admissible

A standard electromagnetic field is admissible if it can be deduced from scalars ϕ, θ, according to (8)–(9), which is not always the case. However, as will be shown in this section, all the standard electromagnetic fields $F_{\mu\nu}$ are locally admissible, except perhaps in a set of points of zero measure. To be specific, this means that the domain D where any solution of the Maxwell equations $F_{\mu\nu}$ is defined is the union of p open subsets D_j plus a zero measure set Σ and in each D_j there are defined scalars ϕ_j, θ_j, so that

$$F_{\mu\nu} = f_{\mu\nu}(\phi_j) \qquad M_{\mu\nu} = f_{\mu\nu}(\theta_j) \tag{31}$$

according to (8)–(9). In other words, this model and Maxwell standard theory (plus the condition $\boldsymbol{E} \cdot \boldsymbol{B} = 0$) are equivalent from the local point of view. Moreover, $F_{\mu\nu}$ can be constructed by patching together the admissible fields defined in D_j.

Let D be a region in three-space, either bounded or not, in which two divergenceless orthogonal vector fields $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{r})$ are defined. If $\boldsymbol{B}$ is admissible, there is a scalar $\phi(\boldsymbol{r})$ so that

$$B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk} \qquad F_{jk} = \frac{1}{2\pi\mathrm{i}}\frac{\partial_j\phi^*\partial_k\phi - \partial_k\phi^*\partial_j\phi}{(1+\phi^*\phi)^2}. \tag{32}$$

$\boldsymbol{E}$ is admissible if it can be expressed in a similar way with a scalar $\theta(\boldsymbol{r})$. If $\phi = S\exp(\mathrm{i}2\pi\sigma)$, F_{jk} can be written as

$$F_{jk} = -\partial_j\frac{1}{1+S^2}\,\partial_k\sigma + \partial_k\frac{1}{1+S^2}\,\partial_j\sigma \tag{33}$$

from which

$$\boldsymbol{B} = \nabla\frac{1}{1+S^2}\times\nabla\sigma. \tag{34}$$

A similar equation holds for $\boldsymbol{E}$.

It is known that any divergenceless vector field can be expressed in terms of the so-called Clebsch variables $\lambda(\boldsymbol{r})$ and $\mu(\boldsymbol{r})$ (Lamb 1932, Kuznetsov and Mikhailov 1980) as

$$\boldsymbol{B} = \nabla\lambda\times\nabla\mu \tag{35}$$

$$\boldsymbol{A} = \lambda\,\nabla\mu + \nabla\xi \tag{36}$$

where the scalar ξ satisfies the Poisson equation

$$\Delta\xi = -\nabla\cdot(\lambda\nabla\mu). \tag{37}$$

However, it should be pointed out that λ and μ are not uniquely defined, many different elections being possible for them. They are two first integrals of the dynamical system

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau} = B_i(\boldsymbol{r}). \tag{38}$$

Both $\nabla\lambda$ and $\nabla\mu$ are single-valued. If λ and μ are also single-valued functions, (38) is integrable and the magnetic lines, which coincide with its trajectories, are the intersections of the surfaces $\lambda(\boldsymbol{r}) = \text{constant}$ and $\mu(\boldsymbol{r}) = \text{constant}$ which foliate the space R^3 and are called accordingly magnetic surfaces. In this case, these lines are not linked and the magnetic helicity vanishes (Kuznetsov and Mikhailov 1980). But, for the system (38) to be integrable and for magnetic surfaces to exist, it is enough that one of them, say λ, be an isolating integral so that the surface $\lambda(\boldsymbol{r}) = \text{constant}$ had only a finite number of branches, because the system operates in these branches and there is no chaos in two-dimensional manifolds. But, if the other variable μ is multiple-valued, the magnetic helicity may be non-zero because of a non-trivial topology of the linked magnetic lines, even if there are still magnetic surfaces. To understand this question, take the case in which $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ in the border ∂D, $\boldsymbol{n}$ being a unit vector normal to this border (Rañada 1992). If μ is single-valued, the helicity

(6) can be written, after integration by parts and application of the Gauss theorem, as

$$\int_D \nabla \cdot (\xi \nabla\lambda \times \nabla\mu)\, \mathrm{d}^3 r = \int_{\partial D} \xi(\nabla\lambda \times \nabla\mu) \cdot \boldsymbol{n}\, \mathrm{d}S$$

which vanishes because of the boundary condition, equation (7) implying then that the linking number of the $\boldsymbol{B}$-lines must be zero. However, if μ is multiple-valued, the argument fails since the Gauss theorem cannot be applied.

Let us consider this problem more closely. If μ is multiple-valued, there will be some curves c_i in R^3 around which it will jump a certain amount $\int \nabla\mu \cdot \mathrm{d}\boldsymbol{r} = \gamma_i \neq 0$, this integral being extended along a path which contours c_i. If $\nabla\mu$ is to be single-valued, this jump must be constant along the curve. The function $\mu(\boldsymbol{r})$ will be isolating if and only if any pair (γ_i, γ_j) is conmensurable, this being equivalent to the statement that a real number q exists such that all the jumps of $\mu' = q\mu$ are integer numbers; in that case, the branches of $\mu'(\boldsymbol{r}) =$ constant do not densely cover any region of the space. Since we can take as Clebsch variables the couple $(\lambda/q, q\mu)$, if $\boldsymbol{B}$ is the vector field of an integrable dynamical system, there is a choice of Clebsch variables such that $\exp(2\pi \mathrm{i}\mu(\boldsymbol{r}))$ is a single-valued function.

It is clear from the previous considerations that $\boldsymbol{B}$ is admissible if there are functions $S(\boldsymbol{r})$ and $\sigma(\boldsymbol{r})$ verifying

$$\frac{1}{1+S^2} = \lambda \qquad \sigma = \mu. \tag{39}$$

If $\boldsymbol{B}$ is integrable in a region D, $\sigma = \mu$ is a good solution, since $2\pi\mu$ has the necessary properties to be the phase of a well defined complex field. On the other hand, a function $S(\boldsymbol{r})$ which satisfies (39) exists only if $0 \leqslant \lambda \leqslant 1$. Now let Σ be the surface in which either $\lambda = \infty$ or $\mu = \infty$. In general, $D - \Sigma$ will have p connected open components D_j. Let $D_j^* \subset D_j$ be open subsets in which λ and μ are bounded. In each one of them we define $\lambda^{(+)} = \sup(\lambda(x))$ and $\lambda^{(-)} = \inf(\lambda(x))$ when $x \in D_j^*$ (subscripts j are omitted here to simplify the notation). The couple

$$(\lambda', \mu') = \left(\frac{\lambda - \lambda^{(-)}}{n}, n\mu\right) \tag{40}$$

is a good choice of the Clebsch variables of $\boldsymbol{B}$ for any integer n in D_j^*. If, furthermore, $n > \lambda^{(+)} - \lambda^{(-)}$, then $0 \leqslant \lambda' \leqslant 1$, so that equations (39) have the solution

$$S = \sqrt{\frac{1-\lambda'}{\lambda'}} \qquad \sigma = \mu' \tag{41}$$

which implies that in any $D_j^* \subset D_j \subset D - \Sigma$, the field can be generated through (32) from the scalar

$$\phi = \sqrt{\frac{n - (\lambda - \lambda^{(-)})}{\lambda - \lambda^{(-)}}}\, \exp(\mathrm{i}2\pi n\mu). \tag{42}$$

It should be emphasized that the volume of $D - \cup D_j^*$ may be made as small as desired. All this means that the field $\boldsymbol{B}(\boldsymbol{r})$ can be obtained by patching together fields $B_j(r)$, each one defined in the corresponding D_j^*, except for a gap $D - \cup D_j^*$ which is as small as required. A similar property holds for the electric field. As a consequence, it can be said that any standard field coincides locally with an admissible field, except in the zero measure set Σ. But it must be stressed that a smooth scalar ϕ (even of class C^1) which generates $B(x)$ in all D does not exist in general.

Now let us consider an electromagnetic field in a vacuum domain D, corresponding to the orthogonal divergenceless Cauchy data $E = E(r)$, $B = B(r)$, at time $t = t_0$. If B is admissible, there exist two functions $\alpha(r), \beta(r)$ such that, by defining the time derivative of the Clebsch variables as

$$\partial_0 \lambda = -\alpha \qquad \partial_0 \mu = \beta \tag{43}$$

the electromagnetic tensor has the expression

$$F_{\alpha\beta} = -\partial_\alpha \lambda \partial_\beta \mu + \partial_\beta \lambda \partial_\alpha \mu \tag{44}$$

so that

$$B = \nabla\lambda \times \nabla\mu \qquad E = -\partial_0 \lambda \nabla\mu + \partial_0 \mu \nabla\lambda. \tag{45}$$

The proof is simple. As $E \cdot B = 0$, E is a linear combination $\alpha\nabla\mu + \beta\nabla\lambda$, so that it suffices to define the time derivative of the Clebsch variables as in (43). Notice that those of the modulus and phase of the scalar ϕ are

$$\partial_0 S = \frac{\partial_0 \lambda}{2\sqrt{\lambda(1-\lambda)}} \qquad \partial_0 \sigma = \partial_0 \mu. \tag{46}$$

In other words: given any regular standard solution of Maxwell equations $F_{\mu\nu}(x)$ at any time t_0, it is locally equal to an admissible one, except perhaps at a zero measure set. To be more precise, we have proved the following proposition.

Proposition. Let a regular electromagnetic field $F_{\mu\nu}(x)$ be given in a region D at a time t_0, with the property $E \cdot B = 0$. Then either it is admissible in D or $D = \cup D_k \cup \Sigma$ the p D_k being disjoint open subregions $k = 1, \ldots, p$, $D_i \cap D_j = \emptyset$ if $i \neq j$, and Σ being the union of their common borders, so that $F_{\mu\nu}(x)$ is admissible in each D_k in the following sense: Given any $\epsilon > 0$, there exist regions $D_k^* \subset D_k$ with the properties:

(i) The volume of $D - \cup D_k^*$ is smaller than ϵ.

(ii) There are $2p$ scalars $\phi_k \theta_k$, $k = 1, \ldots, p$, each pair generating $F_{\mu\nu}(x)$ in the corresponding D_k,

$$F_{\mu\nu}(x) = f_{\mu\nu}(\phi_k) \qquad M_{\mu\nu}(x) = f_{\mu\nu}(\theta_k) \qquad \text{in } D_k.$$

The meaning of this statement is that any standard electromagnetic field in a domain D can be constructed by patching together several admissible fields, each one defined in a subdomain D_k. This implies that, from the local point of view, there is no difference between this model and the standard Maxwell theory for radiation fields, except perhaps in a zero measure set. There is indeed a difference, and it is an important one, but it refers to the global aspects of the solutions and is manifested in the topological structure, as in the existence of a topological constant of the motion.

As a first example, let us consider the plane wave

$$\boldsymbol{E} = \omega E_0\,(0, \sin\omega(x-t), 0) \qquad \boldsymbol{B} = \omega E_0\,(0, 0, \sin\omega(x-t)) \tag{47}$$

which propagates along the Ox-axis. A choice for the Clebsch variables of $\boldsymbol{B}$ is

$$\lambda = E_0\,(1 - \cos\omega(x-t)) \qquad \mu = y. \tag{48}$$

If $n' > 2E_0$, the scalar

$$\phi = \sqrt{\frac{n' - E_0\,(1-\cos\omega(x-t))}{E_0\,(1-\cos\omega(x-t))}}\,\exp(2\mathrm{i}\pi n' y) \tag{49}$$

generates $\boldsymbol{B}(\boldsymbol{r}, t)$, according to (8). It is easy to see that the scalar θ which generates $\boldsymbol{E}(\boldsymbol{r}, t)$ through (9) has the same modulus as ϕ, the phase being instead $2\mathrm{i}\pi n' z$. As we see, (47) is not an admissible field since ϕ and θ do not represent smooth maps $S^3 \to S^2$ because they are not well defined at infinity. However, there are admissible fields which coincide with it in any bounded domain. Or, in other words, it is a local but not a global solution in this model.

Another example is the standing wave with A_μ given by

$$\begin{aligned} &A_0 = 0 \qquad A_1 = A_{01}\cos k_1 x\,\sin k_2 y\,\sin k_3 z\,\cos\omega t \\ &A_2 = A_{02}\sin k_1 x\,\cos k_2 y\,\sin k_3 z\,\cos\omega t \\ &A_3 = A_{03}\sin k_1 x\,\sin k_2 y\,\cos k_3 z\,\cos\omega t \end{aligned} \tag{50}$$

with $\omega^2 = k_i k_i$, which verifies the condition $\boldsymbol{E}\cdot\boldsymbol{B} = 0$. A simple calculation shows that a choice for the Clebsch variables of $\boldsymbol{B}$ is

$$\begin{aligned} &\lambda = 1 + \sin k_1 x\,\sin k_2 y\,\sin k_3 z \\ &\mu = \frac{A_{01}}{k_1}\log|\sin k_1 x| + \frac{A_{02}}{k_2}\log|\sin k_2 y| + \frac{A_{03}}{k_3}\log|\sin k_3 z|. \end{aligned} \tag{51}$$

Accordingly, the magnetic field can be generated by the scalar

$$\phi = \sqrt{\frac{2-\lambda}{\lambda}}\,\exp(4\pi\mathrm{i}\mu)$$

except in the planes $k_1 x = n_1\pi, k_2 y = n_2\pi, k_3 z = n_3\pi$, the n_i being integers, where μ diverges. This means that there are scalars $\phi_{n_1 n_2 n_3}$ defined and smooth in the finite domains $n_1\pi < k_1 x < (n_1+1)\pi$, $n_2\pi < k_2 y < (n_2+1)\pi$, $n_3\pi < k_3 z < (n_3+1)\pi$ which generate the magnetic field in each one of them. However, it cannot be produced by a smooth map $S^3 \to S^2$, because of the singularities of μ. A similar result holds for $\boldsymbol{E}$. Looking from the local point of view, this electromagnetic wave coincides with an admissible field around any point, except in a set of zero measure. However, there is no admissible field which coincides with it throughout all the space R^3.

5. Quantization of the helicity

The term *helicity* can be used in (at least) two different contexts. First, as in this work up to now, it characterizes the topology of the integral lines of a divergenceless field, be it a magnetic or electric field or the vorticity of a fluid. In our case there is a magnetic and an electric helicity, h_{mag}, h_{el} defined by (27) and (28) as

$$h_{\mathrm{mag}} = \int_{R^3} \boldsymbol{A} \cdot \boldsymbol{B}\, \mathrm{d}^3 r \qquad h_{\mathrm{el}} = \int_{R^3} \boldsymbol{C} \cdot \boldsymbol{E}\, \mathrm{d}^3 r. \tag{52}$$

Second, as in elementary particle physics, where it refers to the projection of the spin on the direction of the motion, that is

$$h' = \frac{\boldsymbol{p} \cdot \boldsymbol{s}}{p}. \tag{53}$$

Since the photon helicity h' is quantized and all the electromagnetic fields of this model satisfy the topological quantum condition $h_{\mathrm{mag}} = n$, $h_{\mathrm{el}} = m$, n, m being *integer numbers*, it is difficult avoiding the question: is there any relation between these two types of helicity, h_{mag} and h_{el} on the one side and h' on the other? A positive answer would give a hint that the discreteness of the photon helicity could have a topological basis. Before answering this question, let us established whether there are families of solutions for which $h_{\mathrm{mag}} = h_{\mathrm{el}} = h'$.

Given any two non-trivial maps $\phi, \theta : S^3 \mapsto S^2$, i.e. with non-vanishing Hopf index, and verifying the condition (23), the corresponding electromagnetic fields has a non-vanishing integer magnetic helicity. An interesting example was obtained in a previous paper (Rañada 1990a, 1990b).

Hopf himself proposed the map

$$\phi_{\mathrm{H}}(x, y, z) = \frac{2(x + \mathrm{i} y)}{2z + \mathrm{i}(r^2 - 1)} \tag{54}$$

where $r^2 = x^2 + y^2 + z^2$, as an example in which the index is non-zero. Actually it is equal to one, which means that any two level curves of ϕ_{H} are linked once, as can be checked with the lines $\phi_{\mathrm{H}} = 0$ and $\phi_{\mathrm{H}} = \infty$, the z-axis and the circle $r = 1$, $z = 0$, respectively (see figure 2). Now, let us define two fields ϕ, θ as

$$\phi(x, y, z) = \phi_{\mathrm{H}}(\kappa x, \kappa y, \kappa z) \qquad \theta(x, y, z) = \phi_{\mathrm{H}}^*(\kappa y, \kappa z, \kappa x) \tag{55}$$

where κ is any inverse length, the corresponding Whitehead vectors being

$$B_i = -\tfrac{1}{2}\epsilon_{ijk} f_{jk}(\phi) \qquad E_i = -\tfrac{1}{2}\epsilon_{ijk} f_{jk}(\theta). \tag{56}$$

The solution of the Maxwell equations with Cauchy data at $t = 0$ given by (56) will have any pair of $\boldsymbol{B}$-lines (or of $\boldsymbol{E}$-lines) linked once for all time, since this property is invariant under continuous evolution, so that $h_{\mathrm{mag}} = h_{\mathrm{el}} = 1$. It has been called an *electromagnetic knot* (Rañada 1990a, 1990b). Moreover, since its magnetic lines both turn around the z-axis and the circle $z = 0, r = 1$ and its electric ones do the same around the x-axis and the circle $x = 0, r = 1$, the electromagnetic field can be said to contain a magnetic and an electric vortex, as is schematically indicated in figure 3.

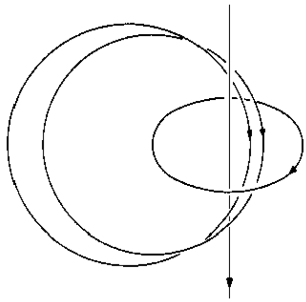

Figure 2. A schematic representation of the Hopf map. Some level curves are shown.

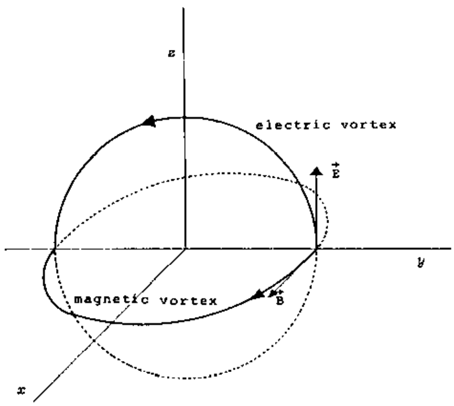

Figure 3. A schematic representation of the Cauchy data of an electromagnetic knot. The electric and magnetic vortexes, which correspond to $\theta = \infty$ and $\phi = \infty$, respectively are shown.

Although these are curious properties, it is nonetheless a standard electromagnetic solution of the Maxwell equations, and this must be stressed.

We may as well generate a solution with magnetic helicity equal to any positive interger n, by following the same steps, but taking the nth power ϕ_H^n instead of ϕ_H. This can seem strange, since the level curves are the same for ϕ as for any of its powers. However, note that, in the case of ϕ^n, the inverse image of any complex number ζ has n determinations $\sqrt[n]{\zeta}$. This means that the degree of the map $\Sigma_a \mapsto S^2$, defined in section 2, must be n. Under reflection with respect to a plane, the magnetic helicity changes its sign, thus giving a procedure to obtain electromagnetic knots with negative helicity.

The energy, momentum and angular momentum of this solution can be computed from the corresponding tensor densities, and their values turn out to be

$$E = 2\kappa \qquad p = (0, \kappa, 0) \qquad J = (0, 1, 0). \tag{57}$$

This angular momentum is in fact the spin because it is taken with respect to the position of the centre of energy $\int rT^{00}\,\mathrm{d}^3r / \int T^{00}\,\mathrm{d}^3r$ at $t = 0$. As it is directed along the direction of p and is equal to one, it is clear that $h' = h_{\mathrm{mag}} = h_{\mathrm{el}} = 1$. This electromagnetic knot is a wavepacket, its centre of energy travelling along the y-axis with velocity $p/E = \frac{1}{2}$. Its Fourier expansion can be easily obtained and is equal to

$$B(r,t) = \frac{\kappa^2}{(2\pi)^{3/2}} \int [R(k)\cos(k\cdot r - \omega t) + R'(k)\sin(k\cdot r - \omega t)]\, d^3k$$

$$E(r,t) = \frac{\kappa^2}{(2\pi)^{3/2}} \int [R'(k)\cos(k\cdot r - \omega t) - R(k)\sin(k\cdot r - \omega t)]\, d^3k \qquad (58)$$

where the vectors $R(k)$ and $R'(k)$ are given by

$$R(k) = \frac{e^{-\omega/\kappa}}{(2\pi)^{1/2}\kappa}\left(\frac{k_1k_3}{\omega}, \frac{k_2k_3}{\omega} + k_3, -\frac{k_1^2+k_2^2}{\omega} - k_2\right)$$

$$R'(k) = \frac{e^{-\omega/\kappa}}{(2\pi)^{1/2}\kappa}\left(\frac{k_2^2+k_3^2}{\omega} + k_2, -\frac{k_1k_2}{\omega} - k_1, -\frac{k_1k_3}{\omega}\right).$$

The vector potential can be chosen as (with $A_0 = 0$)

$$A(r,t) = \frac{\kappa^2}{(2\pi)^{3/2}} \int \frac{1}{\omega}[R(k)\cos(k\cdot r - \omega t) + R'(k)\sin(k\cdot r - \omega t)]\, d^3k.$$

As we see it is a wavepacket peaked at $\omega = \kappa, p = (0, \kappa, 0)$. By Lorentz transforming it with velocity u along the y-axis, a family of electromagnetic knots is obtained, all of which have magnetic helicity equal to one. Their energy, momentum and angular momentum are given by (with $\gamma = 1/\sqrt{1-u^2}$)

$$E' = \gamma(2-u)\kappa \qquad p' = (0, \gamma(1-2u)\kappa, 0) \qquad J' = (0,1,0). \qquad (59)$$

As can be seen, the projection of the spin on the directions of the motion is still $+1$. In other words, all the electromagnetic knots of this family satisfy the equality $h_{\mathrm{mag}} = h_{\mathrm{el}} = h'$, their helicity in the sense of particle physics also being $+1$. Because of this property, they have been called *quasiphotons* (Rañada 1990a, 1990b). It is easy to show that there are other solutions in which the helicities are -1. (Clearly, given a solution with magnetic or electric helicity h, the one obtained by reflection with respect to any plane has helicity $-h$.)

The two meanings of helicity coincide for this family of solutions. Under which conditions does this property hold? The answer is that $h_{\mathrm{mag}} = \pm h'$ if the field verifies the following equality

$$B = \mp i(\mathcal{J}\cdot n)E \qquad (60)$$

where $n = p/p$ is the unit vector in the direction of the linear momentum and $\mathcal{J}$ is the generator of the rotations (i.e. the angular momentum operator). Besides, $h_{\mathrm{el}} = \pm h'$ if

$$E = \pm i(\mathcal{J}\cdot n)B. \qquad (61)$$

The proof is as follows. First note that

$$h' = \int_{R^3} (r \times S)\cdot n\, d^3r \qquad (62)$$

where $S = E \times B$ is the Poynting vector, or otherwise written

$$h' = \int_{R^3} B \cdot ((E \cdot n)r - (E \cdot r)n) \, \mathrm{d}^3 r. \tag{63}$$

Comparing this expression with equations (27) and (28) the announced result follows.

The electromagnetic knots (58) verify

$$\mathrm{i}(\mathcal{J} \cdot n) B = E \qquad \mathrm{i}(\mathcal{J} \cdot n) E = -B. \tag{64}$$

This implies that, under a rotation of $\pi/2$ around the direction of p, the field changes according to $E \to -B$, $B \to E$. As a consequence of (64), all the solutions in this class verify $h_{\mathrm{mag}} = h_{\mathrm{el}} = h'$. There is a second class of fields (obtained from the previous one by reflection with respect to a plane containing the vector p), which verify

$$\mathrm{i}(\mathcal{J} \cdot n) B = -E \qquad \mathrm{i}(\mathcal{J} \cdot n) E = B. \tag{65}$$

all of which verify $h_{\mathrm{mag}} = h_{\mathrm{el}} = -h'$. The reason is that the magnetic helicity changes its sign upon reflection, while the projection of the spin on the linear momentum remains unchanged. In fact, the following has been proved.

Proposition. If condition (64) is satisfied, the helicities verify $h_{\mathrm{mag}} = h_{\mathrm{el}} = h'$. If condition (65) holds, then $h_{\mathrm{mag}} = h_{\mathrm{el}} = -h'$.

Furthermore, if the field can be deduced from the scalars ϕ, θ and is therefore admissible, the three helicities are integer numbers.

6. Quantization of the electric charge

Let $\theta(r, t)$ be a scalar from which the electric field can be deduced as

$$M_{\mu\nu} = f_{\mu\nu}(\theta) = \frac{1}{2\pi \mathrm{i}} \frac{\partial_\mu \theta^* \partial_\nu \theta - \partial_\nu \theta^* \partial_\mu \theta}{(1 + \theta^* \theta)^2}. \tag{66}$$

In other words, the area 2-form in S^2 corresponding to θ is the electric flux 2-form

$$\tfrac{1}{2} f_{ij}(\theta) \mathrm{d}x_i \wedge \mathrm{d}x_j = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}\theta^* \wedge \mathrm{d}\theta}{(1 + \theta^* \theta)^2} \tag{67}$$

$$E_i = -\tfrac{1}{2} \epsilon_{ijk} M_{jk}. \tag{68}$$

Until now, we have been taking complex scalar fields which are regular and have a limit when $r \to \infty$ independent of the direction. With these properties, they represent maps $R^3 \cup \{\infty\} \equiv S^3 \to S^2$ and correspond to electromagnetic fields in vacuum, defined through all R^3. If there is a point charge somewhere, the electromagnetic field is given by the Lienard–Wiechert potential which has the property $E \cdot B = 0$, so that the Faraday 2-form is degenerate which allows this model to be applied. If the point charge be at r_0, the electric field is singular at r_0, its lines diverging at ∞. Consequently, θ is not defined in the sphere S^3, but in S^3 punctured twice, that is in $S^2 \times R$ (note that the electric lines are its level curves). (Intuitively, $R^3 - \{0, \infty\}$

can be understood as the product of the unit sphere $r = 1$ times the semi-straight line $0 < r < \infty$.) It therefore represents a map

$$\theta : S^2 \times R \to S^2. \tag{69}$$

These kind of maps can be classified in homotopy classes labelled by an integer number which is equal to the degree of the map $S^2 \to S^2$. It is easy to understand then that the topological number (the label of the classes of homotopy) is equal to the degree of the map from the sphere $r = 1$ on the complex field C, which has the value

$$\tilde{n} = \text{degree} = \int_{r=1} \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}\theta^* \wedge \mathrm{d}\theta}{(1+\theta^*\theta)^2} = \frac{1}{4\pi} \int_{r=1} \sin\vartheta \, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi \tag{70}$$

where $\sin\vartheta \, \mathrm{d}\theta \wedge \mathrm{d}\varphi$ is the area 2-form in the image sphere $S^2(\equiv C)$. But, at the same time and because of (67) and (68), one has

$$\tilde{n} = \int_{r=1} \tfrac{1}{2}\epsilon_{ijk} E_i \mathrm{d}x_j \wedge \mathrm{d}x_k = \int_{r=1} \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}S. \tag{71}$$

In other words, the electric flux through a sphere which contains the charge is always an integer number $\tilde{n}$. According to the Gauss theorem, this flux is $4\pi q$, from which $q = \tilde{n}/4\pi$, in the natural system of units based on the action constant of the model (see 3.3). In physical units, the charge q is written as

$$q = \tilde{n}\sqrt{a}/4\pi \tag{72}$$

which means that the electric charge is necessarily quantized, as it is always an integer multiple of the fundamental charge

$$q_0 = \sqrt{a}/4\pi. \tag{73}$$

The fact that there is a topological quantization of the electric charge seems encouraging (Rañada 1991b).

We may summarize the topological quantization laws of the helicities and the charge as

$$h_{\mathrm{mag}} = na \qquad h_{\mathrm{el}} = ma \qquad q = \tilde{n} q_0 = \tilde{n}\sqrt{a}/4\pi \tag{74}$$

where n and m are Hopf indexes of maps $S^3 \mapsto S^2$ and $\tilde{n}$ is the degree of a map $S^2 \mapsto S^2$. In order to fix the scale of these quantizations, some criterion for the value of the constant a is required which normalizes the fields. If one takes $a = \hbar$ (the simplest choice), then $h_{\mathrm{mag}} = n\hbar$, $h_{\mathrm{el}} = m\hbar$, $q_0 = e/1.074$, where e is the electron charge and $1.074 = 4\pi/\sqrt{137}$ and if $a = 1.153\hbar$, then $q_0 = e$, but the helicities take 'wrong' values. In any case, in the absence of any reliable criterion for the value of a, these numbers do not have any great significance.

7. Conclusions

The set of the solutions of the Maxwell equations in vacuum contains a subset, called *the admissible fields* here, with a topological structure corresponding to that of the maps $S^3 \mapsto S^2$, between spheres. These maps can be interpreted, by stereographic projection, as complex scalar fields with only one value at infinity. There is one admissible field for each pair of such maps $\phi(\boldsymbol{r}, t)$, $\theta(\boldsymbol{r}, t)$ with orthogonal level curves. Its magnetic and electric lines, respectively, are these level curves. The set of the admissible fields has the following properties.

(i) Their magnetic and electric helicities take only integer values n, m,

$$h_{\text{mag}} = \int_{R^3} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}^3 r = n \qquad h_{\text{el}} = \int_{R^3} \boldsymbol{C} \cdot \boldsymbol{E} \, \mathrm{d}^3 r = m \tag{75}$$

where $\nabla \times \boldsymbol{A} = \boldsymbol{B}$, $\nabla \times \boldsymbol{C} = \boldsymbol{E}$, equal to the Hopf indexes of ϕ and θ. Because of the existence of these topological constants of the motion, the fields are classified in homotopy classes labelled by two integers n, m.

(ii) As both helicities are dimensionally actions, the equations (75) have the form of quantum conditions. Because of them, a classical model based on the admissible fields, has, however, a quantum-like or prequantum character, even if it only makes use of c-numbers,

(iii) Any standard radiation field (with $\boldsymbol{E} \cdot \boldsymbol{B} = 0$) is locally equal to an admissible field, except in a zero measure set, Or, in other words, it can be obtained by patching together admissible fields. This means that the difference between the set of the admissible fields and that of all the solutions of the Maxwell equations in vacuum is not local but global.

(iv) The association of electromagentic fields to maps $S^3 \mapsto S^2$ offers a procedure for constructing *electromagnetic knots*, i.e. solutions of the Maxwell equations in which the magnetic lines (or the electric lines) are linked with the desired linking number. A family of such knots corresponding to $n = m = 1$ was obtained. It turns out that the helicity in the sense used in particle physics, that is the projection of the spin on the direction of the linear momentum, is equal both to the magnetic and to the electric helicity and to 1. A condition for the equality of the helicities was obtained.

(v) In the set of the admissible fields there is a mechanism for the quantization of the electric charge by which the flux through any closed surface which surrounds a charge is an integer number, equal to the degree of a map $S^2 \mapsto S^2$. This corresponds to the existence of a fundamental charge q_0 equal to $\sqrt{a}/4\pi$.

A model based on the admissible fields is not completely realistic since it only contains fields for which the Faraday 2-form is degenerate or, equivalently, radiation fields with orthogonal electric and magnetic fields, so that a generalization is needed. This limitation will be treated in a forthcoming paper. Nevertheless, its topological quantum conditions and its mechanism for the quantization of the electric charge with the fundamental value equal to $\sqrt{a}/4\pi$ suggests that the topological structure on which it is based is worth further study and that it might offer better understanding of the quantization process (Rañada 1991a).

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